DETERMINATION OF EXTERNAL THERMAL LOAD PARAMETERS BY SOLVING THE TWO-DIMENSIONAL INVERSE HEAT-CONDUCTION PROBLEM

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A method is proposed for recovering the space–time behavior of an external unsteady heat load to a cylindrical body by solving the two-dimensional inverse heat-conduction boundary problem.

When one studies the effectiveness of various materials and structures in a high-temperature environment, one is interested in an experimental investigation of the space–time distribution of heat loads for cylindrical bodies. In many cases the only way to determine the thermal boundary conditions is to solve the inverse heat-conduction problem (IHCP). Many papers (see, e.g., the bibliography in [1]) have developed methods and algorithms for solving IHCP, and the results obtained in these have been applied successfully to investigations of unsteady heat and mass transfer. However, in most cases the authors have examined one-dimensional heat-conduction models, which do not always adequately describe the actual heat-transfer processes. One can cite examples of practical investigations of the thermal conditions of various structures in power equipment, aircraft and technical equipment where a need arises for calculated and experimental determination of the heat flux densities and surface temperatures of cylindrical bodies by solving the unsteady two-dimensional IHCP. A method was proposed in [1] of iterative solution of a two-dimensional inverse problem in extreme formulation for bodies of planar form, which can be generalized to body of other form. Below we consider a method and an algorithm for iterative solution of two-dimensional IHCP in the case of a hollow circular cylinder (Fig. 1) and present results of a systematic investigation of this algorithm.

We assume that on the thermally insulated internal surface $r = R_{in}$ of a cylindrical body, we know the temperature $T(\varphi, \tau)$ and need to find the heat flux density supplied to the external surface of the body $r = R$. We shall consider that there is no heat transfer at the boundaries $\varphi=0$ and $\varphi=\varphi_k$. In this formulation the inverse heat-conduction boundary problem is written as follows:

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right), \tag{1}$$

$$R_{in} < r < R,\ 0 < \varphi < \varphi_k,\ 0 < \tau \leqslant \tau_m,$$

$$T(r,\ \varphi,\ 0) = \xi(r,\ \varphi), \tag{2}$$

$$\frac{\partial T(r,\ 0,\ \tau)}{\partial \varphi} = \frac{\partial T(r,\ \varphi_k,\ \tau)}{\partial \varphi} = \frac{\partial T(R_{in},\ \varphi,\ \tau)}{\partial r} = 0, \tag{3}$$

$$T(R_{in},\ \varphi,\ \tau) = f(\varphi,\ \tau). \tag{4}$$

We require to determine the function

$$q_1(\varphi,\ \tau) = -\lambda\frac{\partial T(R,\ \varphi,\ \tau)}{\partial r}. \tag{5}$$

One promising direction in solving inverse heat-transfer problems is to use extremal formulations which include gradient methods for minimizing the discrepancy functional. With these methods one can easily construct algorithms to regularize the solution of incorrectly posed problems [2, 3]. Methods of the type of steepest descent and conjugate gradients are stable with respect to round-off errors, approximations and smoothing [1, 4]. To use these

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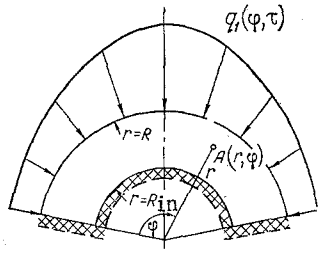

Fig. 1. Solution region for the two-dimensional inverse heat-conduction problem. The thermally insulated boundaries are shown cross-hatched.

one does not require a close initial approximation. They are easily performed on a computer, do not require great machine time, and have the property of a universal approach to solving both linear and nonlinear inverse problems. It is also important to note that, using gradient methods, one can take account of available a priori information on the desired solution, not only qualitative, but also quantitative information.

We shall consider the inverse problem formulated as the problem of seeking a function of two variables

$$P(\varphi, \tau) = -q(\varphi, \tau)/\lambda, \tag{6}$$

giving a minimum of the rms functional

$$J(P) = \int_0^{\tau_m} d\tau \int_0^{\varphi_k} [T(R_{\text{in}}, \varphi, \tau) - f(\varphi, \tau)]^2 d\varphi \tag{7}$$

for the conditions of Eqs. (2)-(4).

One of the central questions in the gradient solution of IHCP, on a successful resolution of which often depends the efficiency of the computational algorithm, is to construct an effective procedure for determining the gradient of the functional (7). This procedure can be obtained by considering the problem conjugate to Eqs. (1)-(4):

$$\frac{\partial \psi}{\partial \tau} = -a\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2}\right), \tag{8}$$

$$R_{\text{in}} < r < R,\ 0 < \varphi < \varphi_k,\ 0 < \tau \leqslant \tau_m, \tag{9}$$

$$\psi(r, \varphi, \tau_m) = 0,$$

$$\frac{\partial \psi(r, 0, \tau)}{\partial \varphi} = \frac{\partial \psi(r, \varphi_k, \tau)}{\partial \varphi} = \frac{\partial \psi(R, \varphi, \tau)}{\partial r} = 0, \tag{10}$$

$$\frac{\partial \psi(R_{\text{in}}, \varphi, \tau)}{\partial r} = 2[T(R_{\text{in}}, \varphi, \tau) - f(\varphi, \tau)]. \tag{11}$$

As the method of minimizing Eq. (7), we choose the method of conjugate gradients, which as the investigations of [1] have shown, has good characteristics for solving incorrectly posed inverse problems, compared with the method of steepest descent. In accordance with this method, we construct an iterative sequence of the type

$$P^{k+1}(\varphi, \tau) = P^k(\varphi, \tau) - \beta_k \xi^k(\varphi, \tau), \tag{12}$$

where $\xi^k(\varphi, \tau)$ is the direction of descent.

The coefficient β_k, which determines the depth of descent in going to the next approximation, is found from the condition

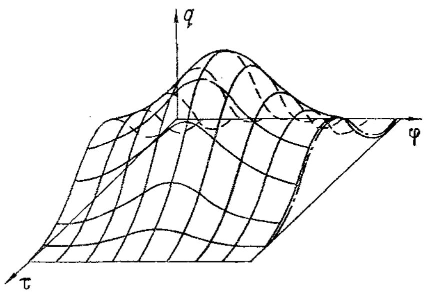

Fig. 2

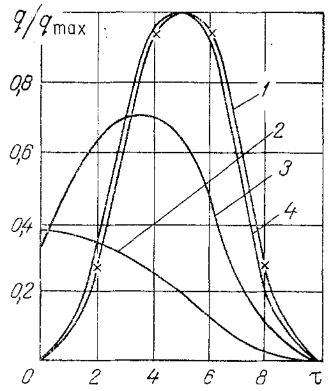

Fig. 3

Fig. 2. Space–time behavior of the external heat load on a cylindrical wall, recorded by solving the two-dimensional inverse heat-conduction problem: the solid lines are accurate values of $q(\varphi, \tau)$; the broken lines are the result of the IHCP solution.

Fig. 3. Illustration of convergence of the iterative process in solving the IHCP with exact original data for $\varphi=90^\circ$ $(0^\circ \leqslant \varphi \leqslant 180^\circ)$: 1) the desired solution; 2) first approximation; 3) ninth approximation; 4) 17th approximation; the points are experimental; τ is in sec.

$$\frac{\partial J(P^{k+1}(\varphi, \tau))}{\partial \beta} = 0. \tag{13}$$

The corresponding optimal value is given by the expression

$$\beta_k = \frac{\int\limits_0^{\tau_m} d\tau \int\limits_0^{\varphi_k} [T(R_{\text{in}}, \varphi, \tau) - f(\varphi, \tau)]\, \Delta T(\xi(\varphi, \tau), \varphi, \tau)\, d\varphi}{\int\limits_0^{\tau_m} d\tau \int\limits_0^{\varphi_k} \Delta T^2(\xi(\varphi, \tau), \varphi, \tau)\, d\varphi}, \tag{14}$$

where the temperature increment $\Delta T(\xi(\varphi, \tau), \varphi, \tau)$ is determined by solving the boundary problem:

$$\frac{\partial \Delta T}{\partial \tau} = a\left(\frac{\partial^2 \Delta T}{\partial r^2} + \frac{1}{r}\frac{\partial \Delta T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Delta T}{\partial \varphi^2}\right), \tag{15}$$

$$R_{\text{in}} < r < R,\ 0 < \varphi < \varphi_k,\ 0 < \tau \leqslant \tau_m, \tag{16}$$

$$\Delta T(r, \varphi, 0) = 0,$$

$$\frac{\partial \Delta T(r, 0, \tau)}{\partial \varphi} = \frac{\partial \Delta T(r, y_k, \tau)}{\partial \varphi} = \frac{\partial \Delta T(R_{\text{in}}, \varphi, \tau)}{\partial r} = 0, \tag{17}$$

$$\frac{\partial \Delta T(R, \varphi, \tau)}{\partial r} = \Delta P(\varphi, \tau). \tag{18}$$

Thus, to solve the two-dimensional IHCP (1)-(4) we have the iterative algorithm (6)-(12), (14)-(18). In accordance with the algorithm considered, to obtain the next approximation we must solve three boundary problems (1)-(4), (8)-(11), and (15)-(18). It is convenient to convert the conjugate problem (8)-(11) to inverse time by introducing the variable $t = \tau_m - \tau$. In this case for all the boundary problems we can use a single computational procedure. On the basis of the above technique for solving a two-dimensional IHCP, we constructed a computational algorithm and wrote a program in Fortran language for the Minsk-32 computer. To solve the boundary problems (1)-(4), (8)-(11), and (15)-(18) in the algorithm, we used the numerical method of variable directions [5]. To investigate the efficiency (accuracy of the results obtained and machine time used) of the proposed method, we conducted numerical experiments on specially chosen model problems. For the numerical solution we

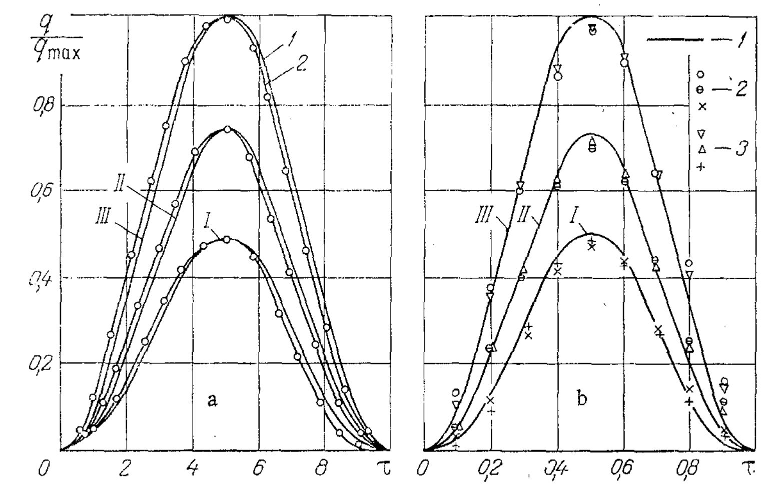

Fig. 4. Results of solving the IHCP at the exact original data (a), and at perturbed but previously smoothed original data (b). For a: 1) exact values of $q(\varphi, \tau)$; 2) 17th approximation; for b: 1) exact values of $q(\varphi, \tau)$; 2) 18th approximation; 3) 19th approximation; I) $\varphi=0°$; 180°; II) $\varphi=45°$; 135°; III) $\varphi=90°$. The quantity τ is in sec.

used various space—time grids, including step changes in Fourier number in the range $0.02 \leqslant \Delta Fo = a\Delta\tau/b^2 \leqslant 2$, where b is the thickness of the cylinder wall. The quantity ΔFo can serve as an independent measure of the effectiveness of the problem solved. It should be noted that for ΔFo < 0.3-0.5 the results of solving this inverse problem with the help of a direct algebraic method, based on the integral form of the IHCP (Chap. 4 in [1]), have a strongly oscillatory character.

Below we present some results of the numerical modeling. For a space mesh $N_r = 40$, $N_\varphi = 40$ and $N_\tau = 40$ the machine time to perform one iteration was 21 min. The numerical experiment showed that decreasing the number of time steps to $N_\tau = 20$ led to loss of accuracy in recovering the boundary conditions. The technical capability (the speed) of the Minsk-32 computer allows a solution to an IHCP for $N_\tau = 60$, and it turned out that for the example considered, an increase in the number of time steps (from $N_\tau = 40$ to $N_\tau = 60$) did not produce a noticeable increase in the accuracy of solving the IHCP. The results of the numerical modeling shown in the figures were obtained for $N_\tau = 40$. Figure 2 shows the form of the model heat load. The convergence of the iterative sequence to the desired solution at the section $\varphi=90°$ $(0° \leqslant \varphi \leqslant 180°)$ is shown in Fig. 3. During the computations performed, we observed no appreciable oscillations in the solution beyond a finite number of approximations. With this behavior of the iterative process, one can shorten the search for a solution, based on comparing two successive approximations according to the condition

$$\max |q^{k+1}(\varphi_n, \tau_m) - q^k(\varphi_n, \tau_m)| \leqslant \varepsilon, \quad n = 1, 2, \ldots, N, \; m = 1, 2, \ldots, M. \tag{19}$$

In the examples considered by using unperturbed original data one can obtain quite an accurate approximation to the desired solution (Fig. 4a). Figure 4b shows an example of recovering the heat flux density using the perturbed original data. Perturbation of the original data (in temperature) was accomplished according to the relation

$$T^*(\varphi, \tau) = T(\varphi, \tau) + \delta_0\omega, \tag{20}$$

where δ_0 is the maximum applied perturbation (in the case considered δ_0 = 5% of T_{max}); ω is a random quantity with a uniform distribution law $(-1 \leqslant \omega \leqslant 1)$.

Before solving the IHCP the initially perturbed original data according to Eq. (20) were smoothed using the procedure of [6], based on the second-order smoothing of Tikhonov [7]. The solution of the IHCP, obtained using the smoothed original data, shows that it is possible in practice to use this method of recovering external boundary conditions.

The results of the numerical modeling allow us to conclude that the accuracy in recovering the heat flux density $q(\varphi, \tau)$ is comparable with that in the original data. The machine time in solving a two-dimensional IHCP proved to be allowable for many practical applications, especially when one uses a high-speed computer of the type EC-1040 or BÉSM-6.

In conclusion, we note that the results obtained can easily be extended to a two-dimensional inverse heat-conduction problem, described in a spherical coordinate system, for the case of an axisymmetric heat load. In addition, the method described may be generalized to a nonlinear formulation of a two-dimensional heat-conduction problem.

NOTATION

a, diffusivity; T, temperature; T*, perturbed values of temperature; τ, time; r, φ, polar three-dimensional coordinates; R_{in}, radius of inside surface of cylindrical wall; R, radius of external surface of cylindrical wall; φ_k, τ_m, largest values of the variables φ and τ; λ, thermal conductivity; q_1, external heat flux density; J, rms functional; ψ, conjugate variable; ξ, direction of descent; β, depth of descent; ΔT, temperature increment; ΔFo, Fourier number step; N_r, N_φ, N_τ, number of steps in space–time mesh; δ_0, maximum value of the perturbation applied at $T(\varphi, \tau)$; ω, random value with uniform distribution law.

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